Lecture 17

Another consequence of DCT is:

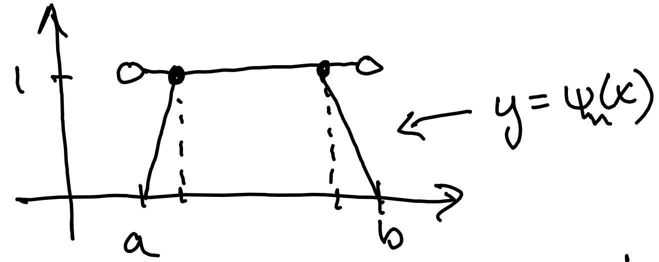
<u>Thm 1</u>. (i) The simple functions are dense in $L^1(X,\mu)$.

(ii) If $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{L}/\mathcal{B}, \mu_F)$, ← Lebesgue-Stieltjes

then the continuous fcns are dense in L^1.

<u>Pf</u>. Recall that dense here means that there are $\varphi_n \in L^1$ (simple in (i), cont. in (ii)) s.t.
$$\|f-\varphi_n\|_{L^1} = \int |f-\varphi_n| \to 0. \qquad (**)$$
For (i), use Thm 2.10 (b) to find simple φ_n s.t. $0 \le |\varphi_1| \le \dots$ and $\varphi_n \to f$ a.e. Moreover, $|f-\varphi_n| \le 2|f|$, so $(**)$ follows by DCT.

For (ii), we first note that by (i) it suffices to approximate simple fcns in L^1. As a first step, we approximate χ_I, where $I=(a,b)$ is a bounded open interval.



In above pic, we have $\|\chi_I - \varphi_n\|_{L^1} \le \mu_F((a, a+\frac{1}{n})) + \mu_F((b-\frac{1}{n}, b))$. Next we recall that $\mu_F((c,d)) = F(d^-) - F(c)$, where $F(d^-) = \lim_{e \to d^-} F(e)$.

It follows that $\|\chi_I - \varphi_n\|_{L^1} \to 0$ as $n \to \infty$.

Now, let $f \in L^1$ and pick $\varepsilon > 0$. Let $\varphi = \sum_{j=1}^{m} c_j \chi_{E_j}$ be a simple fcn in L^1 s.t. $\|f - \varphi\|_{L^1} < \varepsilon/2$.

For each E_j, by reg. props in Thm 1.18, we can find open intervals I_{jk} s.t. $E_j \subseteq \bigcup_{k=1}^{\infty} I_{jk}$ and s.t. $\mu(E_j) \ge \sum_{k=1}^{\infty} \mu(I_{jk}) - \delta_j \varepsilon$, where $\delta_j > 0$ will be chosen shortly. Since each E_j has finite measure ($\varphi \in L^1$), it follows, in particular, that each I_{jk} is an open bdd interval. Let N_j be s.t.

$$\sum_{k=N_j+1}^{\infty} \mu(I_{jk}) \le \delta_j \varepsilon.$$

Let φ be the continuous fcn in L^1

$$\psi = \sum_{j=1}^{m} c_j \sum_{k=1}^{N_j} \varphi_{jk} ,$$

where φ_{jk} are cont. s.t. $\|\chi_{I_{jk}} - \varphi_{jk}\| \leq \frac{\delta_j}{N_j}\varepsilon$.

Then:

$$\|f-\psi\| \leq \|f-\varphi\| + \|\varphi - \psi\| \leq \frac{\varepsilon}{2} + \sum_{j=1}^{m} |c_j| \cdot$$

$$\|\chi_{E_j} - \sum_{k=1}^{N_j} \varphi_{jk}\| \leq \frac{\varepsilon}{2} + \max_j |c_j| \cdot$$

$$\left(\sum_{j=1}^{m} \|\chi_{E_j} - \sum_{k=1}^{N_j} \varphi_{jk}\| \right) \leq \varepsilon/2 + C \cdot$$

$$\sum_{j=1}^{m} \left(\|\chi_{E_j} - \sum_{k=1}^{\infty} \chi_{I_{jk}}\| + \|\sum_{k=1}^{N_j} \chi_{I_{jk}} - \sum_{j=1}^{N_j} \varphi_{jk}\| + \right.$$

$$\left. \|\sum_{k=N_j+1}^{\infty} \chi_{I_{jk}}\| \right) \leq \frac{\varepsilon}{2} + C \sum_{j=1}^{m} \left(\delta_j \varepsilon + \sum_{k=1}^{N_j} \frac{\delta_j}{N_j} \varepsilon \right.$$

$$\left. + \delta_j \varepsilon \right) = \frac{\varepsilon}{2} + 3C \cdot \left(\sum_{j=1}^{m} \delta_j \right) \varepsilon .$$ Thus,

let now $3\sum_{j=1}^{m} \delta_j = \frac{1}{2}$. Then, we have

$\|f-\psi\| < \varepsilon$ and we conclude that the continuous fcns are dense in $L^1(\mathbb{R}, \mu_F)$. $\boxtimes$